

Time of day	Mosquito species		
	A	B	C
Mid morning	0.0088	15.4900	5.5000
Mid Afternoon	1.3400	0.0300	0.8700
Dusk	0.2600	5.4200	3.2000

The Chi square test statistic is the sum of the chi square values in each cell.

$$\sum \frac{(\text{Observed} - \text{Expected})^2}{\text{Expected}} = 32.12335 \quad \text{with d. f.} = (\text{rows}-1) * (\text{col}-1) = 2 * 2 = 4$$

### The steps of the hypothesis test

- 1) Mosquito species occurrence is independent of time of day
- 2) Mosquito occurrence is NOT independent
- 3) Assume IID r.v.
- 4)  $\alpha = 0.05$ , the critical value with  $(r-1)(c-1) = 4$  d.f. for  $\alpha = 0.05$  is 9.4877.
- 5) From the sample we get a calculated  $\chi^2 = 32.12335$
- 6) The calculated test statistic exceeds the tabular value with  $(r-1)(c-1) = 4$  d.f.. The critical tabular values are;

$$\alpha = 0.05, \quad \chi^2 = 9.4877 \quad \text{statistically significant}$$

$$\alpha = 0.01, \quad \chi^2 = 13.2767 \quad \text{highly significant}$$

$$\alpha = 0.001, \quad \chi^2 = 18.4662 \quad \text{Wow!}$$

Using this terminology we see we have highly significant departure in this case.

So what is the “P value”, the chance of finding a value of 32.1335 or greater by random chance? This is  $P(>\chi^2) = 0.0000018052$ , about 2 in a million.

- 7) So we conclude that the occurrence of mosquito species is not independent of the time of day. It appears to be very dependent.

### SAS example 3a of the Chi square Test of Independence in SAS

Two varieties of a particular moth species occur in two colors (brown and white). A biologist in North Carolina wants to know if the distribution of the two varieties varies with the area of the state. He collects individuals from each region of the state and note the number of each variety.

See computer output

## Chi Square Goodness of Fit

This test is similar to the Chi Square test of independence, but instead of deriving the expected values from the row and column sums, the expected values are derived from some theoretical distribution.

In a flowering plant with incomplete dominance a cross between a red flowered parent (RR) and a white flowered parent (rr) is expected to yield offspring with pink flowers (Rr). If the offspring are then crossed with each other, a ratio of 9:6:1 or red to pink to white flowered offspring should result. In one particular experiment the offspring produced the observed results of 153 red, 72 pink and 17 white plants. Does this result conform to the expected results?

- 1)  $H_0$ : Results follow the expected proportions
- 2)  $H_1$ : Results do not follow the expected proportions
- 3) Assume IID r.v.
- 4) Set  $\alpha$  (at say 0.05 or 0.01 as before). Calculate the critical value where the chi square results for this test have  $c-1 = 2$  degrees of freedom (where  $c$  is the number of column or the number of categories, 3 in this case). The critical value for  $\alpha = 0.05$  is 5.991 and for  $\alpha = 0.01$  it is 9.210.
- 5) Conduct an experiment to obtain results. We got 242 flowers from our experiment.

	Red	Pink	White	Total
Expected ratio	9	6	1	16
Expected proportions	0.5625	0.3750	0.0625	
Observed numbers	153	72	17	242
Expected numbers	136.13	90.75	15.13	
Chi square	2.09	3.87	0.23	6.2

- 6) Compare the calculated chi square value to the test statistic. The calculated value (6.2) does exceeds the test statistic value of  $\alpha = 0.05$ , but not for  $\alpha = 0.01$ .

Tabular Chi square values	alpha	Chi square value
$n = 3$	0.1	4.605
d.f. = 2	0.0500	5.991
	0.0100	9.210
	0.0010	13.815
Actual $P(>\chi^2)$	0.0451	

- 7) In this case we reject the null hypothesis and might term our result as a “statistically significant” departure from the expected result under the null hypothesis. As a published result we may wish to state that the P value was 0.0451, clearly indicating that our result was significant at the 0.05 level of  $\alpha$ , but not at a higher level.

### SAS example 3b of the Chi square Goodness of Fit test in SAS

See computer output

### SAS example 3c – testing a simple ratio with Chi square

See computer output

A final note on the Chi square tests of hypothesis.

Although tests of hypothesis about variances may be either directional or non-directional, the chi square tests of independence are directional. Small values of the chi square statistics for these tests indicate that the null hypothesis is met very well since the observed values are very close to the expected values. It is only with larger departures of the expected values from the observed values that the chi square statistic should be rejected. Therefore, it is only excessively large chi square values that cause the null hypothesis to be rejected and this would be a one tailed test.

## Summary

### The Chi square distribution

The Chi square distribution can be derived as the square of the Z distribution.

Sample variances are Chi square distributed

The Chi square distribution can be used to test hypotheses about variances.

The distribution has only one parameter,  $\gamma$ , (and is different for every  $\gamma$ ) the distribution is non-negative and asymmetrical. The variance of the distribution is  $2\gamma$ .

Hypothesis testing employs the form of the distribution  $\chi^2 = SS / \sigma_0^2$

For testing variance we assume the variable  $Y_i$  is a Normally and Independently distributed random variable (NID r.v.)

### In the Chi square tables

Degrees of freedom are on the left and a different distribution is given in each row.

Selected probabilities in the upper TAIL of the distribution is given in the row at the top of the table.

The distribution is NOT symmetric, so the probabilities at the top must be used for both upper and lower limits.

### In addition to tests of variances, the Chi square can be used to do

Test of Independence

Test of Goodness of Fit

For these tests we assume  $Y_i$  is an Identically and Independently distributed random variable (IID r.v.)

In SAS the Chi square test of independence can be done with PROC FREQ.

## The F test

This test can be used to either,

test the equality of population variances – our present topic

test the equality of population means (this will be discussed later under ANOVA)

The F test is the ratio of two variances (the ratio of two chi square distributions)

### Given two populations

	Population 1	Population 2
Mean	$\mu_1$	$\mu_2$
Variance	$\sigma_1^2$	$\sigma_2^2$

### Draw a sample from each population

	Population 1	Population 2
Sample size	$n_1$	$n_2$
d.f.	$\gamma_1$	$\gamma_2$
Sample mean	$\bar{Y}_1$	$\bar{Y}_2$
Sample variance	$S_1^2$	$S_2^2$

### To test the Hypothesis

$$H_0: \sigma_1^2 = \sigma_2^2$$

$$H_1: \sigma_1^2 \neq \sigma_2^2 \text{ (directional, or one sided, hypotheses are also possible)}$$

The test statistic is  $F = \frac{\sigma_1^2}{\sigma_2^2}$  which has an expected value of 1 under the null hypothesis. In practice there will be some variability, so we need to define some reasonable limits and this will require another statistical distribution.

## The F distribution

1) The F distribution is another family of distributions, each specified by a PAIR of degrees of freedom,  $\gamma_1$  and  $\gamma_2$ .

$\gamma_1$  is the d. f. for the numerator

$\gamma_2$  is the d. f. for the denominator

Note: the two samples do not have to be of the same size and usually are not of the same size.

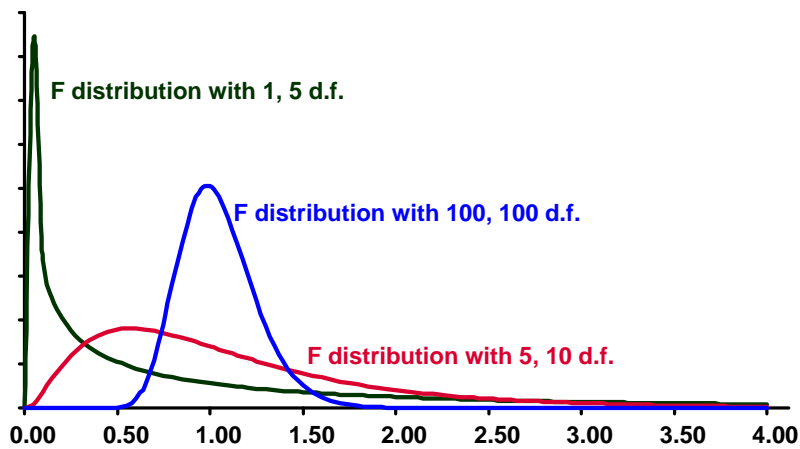
2) The F distribution is an asymmetrical distribution with values ranging from 0 to  $\infty$ , so  $[0 \leq F \leq \infty]$ .

3) There is a different F distribution for every possible pair of degrees of freedom.

4) In general, an F value with  $\gamma_1$  and  $\gamma_2$  d.f. is not the same as an F value with  $\gamma_2$  and  $\gamma_1$  d.f., so order is important. i.e.  $F_{\gamma_1, \gamma_2} \neq F_{\gamma_2, \gamma_1}$  usually

5) The expected value of any F distribution is 1 if the null hypothesis is true.

The F distribution



### The F tables

- The numerator d.f. ( $\gamma_1$ ) are given along the top of the page, and the denominator d.f. ( $\gamma_2$ ) are given along the left side of the page.
- Some tables give only one F value at each intersection of  $\gamma_1$  and  $\gamma_2$ . The whole page would be for a single a value and usually several pages would be given.
- Our tables will give four values at the intersection of each  $\gamma_1$  and  $\gamma_2$ , each for a different a value. These a values are given in the second column from the left.
- Our tables will have two pages.
- Only a very few probabilities will be available, usually 0.05, 0.025, 0.01 and 0.005, and sometimes 0.100.
- Only the upper tail of the distribution is given.

### Partial F table

d.f.	P>F	1	2	3	4	5	10	$\infty$
1	0.05	161	199	216	225	230	242	254
	0.025	648	799	864	900	922	969	1018
	0.010	4052	4999	5404	5624	5764	6056	6366
	0.005	16212	19997	21614	22501	23056	24222	25466
2	0.05	18.5	19.0	19.2	19.2	19.3	19.4	19.5
	0.025	38.5	39.0	39.2	39.2	39.3	39.4	39.5
	0.010	98.5	99.0	99.2	99.3	99.3	99.4	99.5
	0.005	198.5	199.0	199.2	199.2	199.3	199.4	199.5
3	0.05	10.13	9.55	9.28	9.12	9.01	8.79	8.53
	0.025	17.44	16.04	15.44	15.10	14.88	14.42	13.90
	0.010	34.12	30.82	29.46	28.71	28.24	27.23	26.13
	0.005	55.55	49.80	47.47	46.20	45.39	43.68	41.83

## Working with F tables

The F tables are used in a fashion similar to other statistical tables.

Select a probability value at the intersection of the two degrees of freedom.

Be sure to keep track of which one is the numerator degrees of freedom (top of table) and which is the denominator degrees of freedom (left side of table).

At the intersection of the degrees of freedom there are four F values corresponding to  $\alpha$  values of 0.05, 0.25, 0.10 and 0.005.

Find the corresponding F value for the desired  $\alpha$ .

### Example of F table use, one tailed example:

Find F with (5,10) d.f. for  $\alpha = 0.05$

find  $F_{0.05}$  such that  $P[F \geq F_{0.05, 5, 10 \text{ d.f.}}] = 0.050$  where  $\gamma_1 = 5$  and  $\gamma_2 = 10$

For  $F_{5, 10 \text{ d.f.}}$  The tabular values are listed as 3.33, 4.24, 5.64, 6.87

These represent

$$P[F_{5, 10 \text{ d.f.}} \geq 2.52] = 0.100 \text{ (Not in your table)}$$

$$P[F_{5, 10 \text{ d.f.}} \geq 3.33] = 0.050$$

$$P[F_{5, 10 \text{ d.f.}} \geq 4.24] = 0.025$$

$$P[F_{5, 10 \text{ d.f.}} \geq 5.64] = 0.010$$

$$P[F_{5, 10 \text{ d.f.}} \geq 6.87] = 0.005$$

so the value we are looking for is 3.33. For this value  $P[F_{5, 10 \text{ d.f.}} \geq 3.33] = 0.050$ .

Note that since this is a 1 tailed value, then  $P[F_{5, 10 \text{ d.f.}} \leq 3.33] = 0.950$ , so the two sides sum to 1

Also note that if we reverse the d.f., the table shows that  $P[F_{10, 5 \text{ d.f.}} \geq 4.74] = 0.050$ , so the F values generally differ when d.f. are reversed

### More working with F tables

Only the upper tail of the distribution is given. There are three reasons for this.

- Most F tests, including those for Analysis of Variance (ANOVA), are one tailed tests, where the lower tail is not needed.
- The need to calculate the lower tail can be eliminated in some two-tailed cases.
- The value of F for the lower tail can be found by transformation of values from the upper tail.

### Calculating lower tail values for the F distribution

To obtain the lower tail for a value F  $\gamma_1, \gamma_2$  for a particular value of  $\alpha$

First obtain the value in the upper tail for F  $\gamma_2, \gamma_1$  for the same value of  $\alpha$  (note the change in order of the d.f.)

Then calculate  $1/F_{\gamma_2, \gamma_1}$  to get the lower tail.

F table : two-tailed example

Find both upper and lower limits for F with (8, 10) d.f. for  $\alpha = 0.05$

Find  $F_{\alpha/2, 8, 10}$  lower and  $F_{\alpha/2, 8, 10}$  such that

$$P[F_{0.975, 8, 10} \leq F \leq F_{0.025, 8, 10}] = 0.950 \text{ where } \gamma_1 = 8 \text{ and } \gamma_2 = 10$$

For the upper tail, the value we are looking for can be read directly from the table. It is 3.85.

$$P[F_{8, 10} \geq 3.85] = 0.025$$

note that we use only  $\alpha/2$  as the probability for one of the two tails

To find the lower tail, we reverse the d.f., we find that

$$P[F_{10, 8} \geq 4.30] = 0.025 \text{ and then calculate}$$

$$F_{8, 10 \text{ d.f. lower limit}} = 1/F_{10, 8} = 1/4.30 = 0.2326$$

Note the reversal of the order of the degrees of freedom

### Numerical example of an F-test of hypothesis

The concentration of blue green algae was obtained for 7 phytoplankton-density samples taken from each of two lake habitats. Determine if there is a difference in the variability of phytoplankton density between the two habitats.

- 1)  $H_0: \sigma_1^2 = \sigma_2^2$
- 2)  $H_1: \sigma_1^2 \neq \sigma_2^2$
- 3) Assume: Independence (randomly selected samples) and that BOTH populations are normally distributed.
- 4)  $\alpha = 0.05$  and the critical limit is

Calculate

$$P[F_{\text{lower}} \leq F \leq F_{\text{upper}}] = 1 - P[F_{\text{lower}} \leq F] - P[F \geq F_{\text{upper}}] = 1 - 0.025 - 0.025 = 0.95$$

$$P[F \geq F_{\text{upper}}] = 0.025 \text{ we can get directly from the table, } F_{\alpha=0.025, 6, 6} = 5.82$$

$$P[F_{\text{lower}} \leq F] = 0.025 \text{ we calculate as}$$

$$1/P[F \geq F_{\text{upper}}] = 1/5.82 = 0.1718$$

$$P[0.1718 \leq F \leq 5.82] = 0.95$$

This case is uncommon because d.f. upper = d.f. lower, so the F values are the same.

- 5) Draw a sample of 7 from each habitat; calculate the variances and the F ratio.

**Sample data**

	Habitat 1	Habitat 2
Observation 1	7.6	5.9
Observation 2	0.4	3.8
Observation 3	1.1	6.5
Observation 4	3.2	18.3
Observation 5	6.5	18.2
Observation 6	4.1	16.1
Observation 7	4.7	7.6

**Summary statistics**

Statistic	Habitat 1	Habitat 2
$\Sigma Y_i$	27.6	76.4
$\Sigma Y_i^2$	150.52	1074.6
SS	41.70	240.75
$\gamma$	6	6
$S^2$	6.95	40.12
S	2.64	6.33
Mean ( $\bar{Y}$ )	3.94	10.91

Then calculate the F value as  $F = \frac{S_1^2}{S_2^2} = \frac{6.95}{40.12} = 0.1732$

6) Compare the calculated value (0.1732) to the critical region. Given  $\alpha = 0.05$  and a TWO TAILED alternative, and knowing that the degrees of freedom are  $\gamma_1 = 6$  and  $\gamma_2 = 6$ , (note that both are equal), the critical limits are  $P[0.1718 \leq F \leq 5.82] = 0.95$ . Since our calculated F value is between these limit values we would fail to reject the null hypothesis, concluding that the data is consistent with the null hypothesis.

But it was close. Maybe there is a difference and we did not have enough power.

**Some notes on F tests**

NOTE that in this example the smaller value fell in the numerator. As a result, we were comparing the F value to the lower limit.

However, for two tailed tests, it makes no difference which falls in the numerator, and which in the denominator. As a result, we can ARBITRARILY decide to place the larger value in the numerator, and compare the F value to the upper limit.

The need to calculate the lower limit can be eliminated if we calculate  $F = \frac{S_{larger}^2}{S_{smaller}^2}$ .

However, don't forget that this arbitrary placing of the larger variance estimate in the numerator is done for TWO TAILED TESTS ONLY, and therefore we want to test against  $F_{\alpha/2}$ .

There are three common cases in F testing (actually two common and one not so common).

1) Frequently, particularly in ANOVA (to be covered later), we will test  $H_0: \sigma_1^2 = \sigma_2^2$  against the alternative,  $H_1: \sigma_1^2 > \sigma_2^2$ . In this case we ALWAYS form the F value as  $F = \frac{S_1^2}{S_2^2}$ .



**We put the variance that is expected to be larger in the numerator for a one tailed test!**

Don't forget that this is one tailed and all of  $\alpha$  is placed in the upper tail. In the event that  $F < 1$  we don't even need to look up a value in the table, it cannot be "significant".

2) Normal 2 tailed tests (used in 2 sample t-tests to be covered later) will test  $H_0: \sigma_1^2 = \sigma_2^2$  against

the alternative,  $H_1: \sigma_1^2 \neq \sigma_2^2$ . Here we can form the F value as  $F = \frac{S_{\text{larger}}^2}{S_{\text{smaller}}^2}$ .

Don't forget that this is a 2-tailed test and it is tested against the upper tail with only half of  $\alpha$  (i.e.  $\alpha/2$ ) in the upper tail.

When the larger value is placed in the numerator there is no way that we can get a calculated  $F < 1$ .

3) If both the upper and lower bounds are required (not common, found mostly on EXAMS in basic statistics) then we will be testing  $H_0: \sigma_1^2 = \sigma_2^2$  against the alternative  $H_1: \sigma_1^2 \neq \sigma_2^2$ . We can form the F value any way we want, with either the larger or smaller variance in the numerator.

This is a 2 tailed test with  $\alpha/2$  in each tail, and F can assume any positive value (0 to  $\infty$ )

**Summary**

The F distribution is ratio of two variances (i.e. two Chi square distributions) and is used test used to test two variances for equality. The null hypothesis is  $H_0: \sigma_1^2 = \sigma_2^2$ .

The distribution is an asymmetrical distribution with values ranging from 0 to  $\infty$ , and an expected value of 1 under the null hypothesis.

The F tables require two d.f. (numerator and denominator) and give only a very few critical values.

Many, perhaps most, F tests will be directional. For the tests the variance that is expected to be larger and hypothesized to be larger goes in the numerator whether it is actually larger or not. This value is tested against the upper tail with a probability equal to  $\alpha$ .

For the non-directional alternative we may arbitrarily place the larger variance in the numerator and test against the upper tail, but don't forget to test against  $\alpha/2$ .

**Probability Distribution interrelationships**

The probability tables that we have been examining are interrelated. One of these interrelationships is actually pretty important!

If you examine the F table it turns out that in addition to F values the first column is equal to values of  $t^2$ , the last value in the first column corresponds to a  $Z^2$  and the last row is a Chi square value divided by degrees of freedom.

d.f	1	2 ...	10	...	$\infty$
1 2 . . . 10 . . .	$t^2$ (two tailed)  (4.96)	F values		more F values	
$\infty$	$Z^2$ (3.84)		$\gamma_1=10$ $\square$ 1.83		1

The distributions and relationships we have discussed are:

- 1)  $Z_i = \frac{Y_i - \mu}{\sigma}$  for observations and  $Z = \frac{\bar{Y} - \mu_0}{\sigma_{\bar{Y}}} = \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}}$  for testing hypothesis about means.
- 2)  $\chi^2 = Z^2$  with 1 d.f.  
 $\chi^2 = \sum Z^2$  with n d.f.  
 $\chi^2 = SS/\sigma^2$  with n-1 d.f.
- 3)  $t = \frac{\bar{Y} - \mu}{S_{\bar{Y}}} = \frac{\bar{Y} - \mu}{S/\sqrt{n}}$  with n-1 d.f.
- 4)  $F = \frac{S_1^2}{S_2^2}$  with  $n_1-1, n_2-1$  d.f.

**Interrelationships**

- 1)  $\chi^2/\gamma$  with  $\gamma$  d.f. = F with  $\gamma, \infty$  d.f.

$$\chi^2/\gamma = \left( \frac{SS/\sigma^2}{\gamma} \right) = \left( \frac{SS/\sigma^2}{\gamma} \right) \left( \frac{1/\gamma}{1/\gamma} \right) = \left( \frac{SS/\gamma\sigma^2}{1/\gamma} \right) = \left( \frac{SS/\gamma}{\sigma^2} \right) = S^2/\sigma^2$$

which follows an F distribution with  $\gamma, \infty$  d.f.

$$2) F = \frac{\chi_1^2/\gamma_1}{\chi_2^2/\gamma_2} \text{ with } \gamma_1, \gamma_2 \text{ d.f. if } H_0 \text{ is true}$$

given  $\chi^2/\gamma = S^2/\sigma^2$  from part 1 above

then  $\chi^2 = \gamma S^2 / \sigma^2$  and  $\sigma^2 \chi^2 = \gamma S^2$  and  $S^2 = \sigma^2 \chi^2 / \gamma$  with  $\gamma$  d.f.

therefore,  $F = S_1^2 / S_2^2 = \sigma_1^2 \chi_1^2 / \gamma_1 / \sigma_2^2 \chi_2^2 / \gamma_2$  with  $n_1 - 1, n_2 - 1 = \gamma_1, \gamma_2$  d.f.

if  $H_0$  is true, then  $\sigma_1^2 = \sigma_2^2$ , then  $F = \chi_1^2 / \gamma_1 / \chi_2^2 / \gamma_2$  with  $\gamma_1, \gamma_2$  or  $n_1 - 1, n_2 - 1$  d.f.

3)  $t$  with  $\gamma = \infty$  follows a  $Z$  distribution, since as  $\gamma$  increases the sample variance ( $S^2$ ) approaches the population variance ( $\sigma^2$ ). That is, as the sample size approaches infinity the  $t$  distribution,

$$t = \frac{(Y_i - \bar{Y})}{S} \text{ approaches the } Z \text{ distribution } Z = \frac{(Y_i - \mu)}{\sigma}.$$

4)  $Z^2 = F$  with 1,  $\infty$  d.f.

we saw that  $Z^2 = \chi^2$  with 1 d.f.

we saw that  $\chi^2 / \gamma = F$  with  $\gamma, \infty$  d.f.

then  $F = \chi^2 / \gamma = Z^2 / \gamma = Z^2 / 1 = Z^2$ .

5)  $t^2$  with  $\gamma$  d.f. =  $F$  with 1,  $\gamma$  d.f.

This can be shown in several ways. First, we just saw that  $Z^2 = F$  with 1,  $\infty$  d.f. This suggests that  $t^2 = F$  with 1,  $\gamma$  d.f. Another type of proof is given below.

$$F = S_1^2 / S_2^2 \text{ with } \gamma_1, \gamma_2 \text{ d.f.}$$

recall,  $SS = \sum (Y_i - \bar{Y})^2$  with  $n$  d.f., and where  $\hat{Y}_i$  is the mean of a subgroup of the data partitioned into two (or more) groups (the basis of Analysis of Variance).

$$SS = \sum (Y_i - \bar{Y})^2 = \sum_{i=1}^n (Y_i - \bar{Y} + \hat{Y}_i - \hat{Y}_i)^2 = \sum_{i=1}^n ((\hat{Y}_i - \bar{Y}) + (Y_i - \hat{Y}_i))^2$$

d.f are  $n = (n - 1) + 1$

Let  $S_1^2 = n(\bar{Y} - \mu)^2$  with 1 d.f.

$$\text{Let } S_2^2 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{n-1} \text{ with } n-1 \text{ d.f.}$$

both of which are unbiased estimates,

$$\text{then } F = S_1^2 / S_2^2 = \frac{n(\bar{Y} - \mu)^2}{S_2^2} \text{ with } 1, n-1 \text{ d.f.}$$

$$\text{and } F = \frac{n(\bar{Y} - \mu)^2}{S_2^2} = \frac{(\bar{Y} - \mu)^2}{S_2^2 / n} = \left( \frac{(\bar{Y} - \mu)}{S_2 / \sqrt{n}} \right)^2 = t^2 \text{ with } n-1 \text{ d.f.}$$

## Summary

- 1)  $\chi^2/\gamma$  with  $\gamma$  d.f. = F with  $\gamma, \infty$  d.f.
- 2)  $F = \frac{\chi_1^2/\gamma_1}{\chi_2^2/\gamma_2}$  with  $\gamma_1, \gamma_2$  d.f. if  $H_0$  is true
- 3) t with  $\gamma = \infty$  follows a Z distribution
- 4)  $Z^2 = F$  with 1,  $\infty$  d.f.
- 5)  **$t^2$  with  $\gamma$  d.f. = F with 1,  $\gamma$  d.f.**

Some Examples in the F tables (all  $\alpha = 0.05$ , two tails for Z and t values since sign is lost in squaring)

$$F \text{ with } 1, 10 \text{ d.f.} = 4.96 = t^2 \text{ with } 10 \text{ d.f.} = (2.228)^2 = 4.96$$

$$F \text{ with } 1, \infty \text{ d.f.} = 3.84 = Z^2 = (1.96)^2 = 3.84$$

$$F \text{ with } 10, \infty \text{ d.f.} = 1.83 = \chi^2 / \gamma = 18.3 / 10 = 1.83 \text{ with } 10 \text{ d.f.}$$

## Confidence intervals and margin of error

The confidence interval is an expression of what we believe to be a range of values that is likely to contain the true value of some parameter is called a confidence interval. The width of this interval above and below the parameter estimate is called the margin of error.

We can calculate confidence intervals for means ( $\mu$ ) and variances ( $\sigma^2$ ).

## Confidence intervals for t and Z distributions

t and Z distribution confidence intervals start with a t or Z probability statement.

$$P(-t_{\alpha/2} \leq t \leq t_{\alpha/2}) = 1 - \alpha \text{ can also be written}$$

$$P(-t_{\alpha/2} \leq \frac{\bar{Y} - \mu}{S_{\bar{Y}}} \leq t_{\alpha/2}) = 1 - \alpha$$

which is modified to express an interval about  $\mu$  instead of t (or Z).

$$P(-t_{\alpha/2} S_{\bar{Y}} \leq \bar{Y} - \mu \leq t_{\alpha/2} S_{\bar{Y}}) = 1 - \alpha$$

$$P(-\bar{Y} + t_{\alpha/2} S_{\bar{Y}} \geq -\mu \geq -\bar{Y} - t_{\alpha/2} S_{\bar{Y}}) = 1 - \alpha$$

The final form is given below.

$$P(\bar{Y} - t_{\alpha/2} S_{\bar{Y}} \leq \mu \leq \bar{Y} + t_{\alpha/2} S_{\bar{Y}}) = 1 - \alpha$$

The expression for Z has an identical derivation.

$$P(\bar{Y} - Z_{\alpha/2} \sigma_{\bar{Y}} \leq \mu \leq \bar{Y} + Z_{\alpha/2} \sigma_{\bar{Y}}) = 1 - \alpha$$

A common short notation for the interval in the probability statement is given as  $\bar{Y} \pm t_{\alpha/2} S_{\bar{Y}}$ , but the probability statement is preferable as a final result. The value  $t_{\alpha/2} S_{\bar{Y}}$  for intervals on means and  $t_{\alpha/2} S$  for intervals on individual observations is half of the interval width from the lower limit to the upper limit and is called the margin of error.

### Confidence intervals for variance

Variances follow a Chi square distribution. The confidence interval for variance is based on the Chi Square distribution.

$$P(\chi_{lower}^2 \leq \chi^2 \leq \chi_{upper}^2) = 1 - \alpha \text{ or}$$

$$P(\chi_{lower}^2 \leq \frac{SS}{\sigma^2} \leq \chi_{upper}^2) = 1 - \alpha$$

which is solved to isolate  $\sigma^2$ .

$$P\left(\frac{1}{\chi_{lower}^2} \geq \frac{\sigma^2}{SS} \geq \frac{1}{\chi_{upper}^2}\right) = 1 - \alpha$$

$$P\left(\frac{1}{\chi_{upper}^2} \leq \frac{\sigma^2}{SS} \leq \frac{1}{\chi_{lower}^2}\right) = 1 - \alpha$$

giving the expression,

$$P\left(\frac{SS}{\chi_{upper}^2} \leq \sigma^2 \leq \frac{SS}{\chi_{lower}^2}\right) = 1 - \alpha$$

Notice that the upper tabular Chi square value comes out in the lower bound and the lower Chi square in the upper bound.

$$P\left(\frac{SS}{\chi_{upper}^2} \leq \sigma^2 \leq \frac{SS}{\chi_{lower}^2}\right) = 1 - \alpha$$

### Notes on confidence intervals

One sided intervals are possible, but uncommon.

Confidence intervals are one of the most common expressions in statistics, frequently occurring in publications.

Margins of error and confidence intervals are not always calculated in statistical software programs, but they can easily be done by hand.

### From the previous SAS Example 2c

We receive a shipment of apples that are supposed to be “premium apples”, with a diameter of at least 2.5 inches. We will take a sample of 12 apples, and place a confidence interval on the mean. The sample values for the 12 apples are;

2.9, 2.1, 2.4, 2.8, 3.1, 2.8, 2.7, 3.0, 2.4, 3.2, 2.3, 3.4